University of California, Berkeley
Physics 110A Fall 2001   Section 1 (*Strovink*)

SOLUTION TO MIDTERM EXAMINATION 2

Directions: Do both problems, which have equal weight. This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. A photocopy of the four inside covers of Griffiths is included with the exam. Calculators are not needed, but you may use one if you wish. Laptops and palmtops should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (50 points)
A cylindrically symmetric region is bounded by $-\infty < z < \infty$ and $s < s_0$ (s is the cylindrical radius in Griffiths' notation). Within this region, the magnetic field may be obtained from the vector potential

$$\mathbf{A}(s) = \hat{z}\mu_0 C s^2 \ ,$$

where C is uniform, *i.e.* independent of $\mathbf{r}$. (You don't need to choose a particular gauge in order to work this problem, but, if it is helpful, you may work in Lorentz gauge $\nabla \cdot \mathbf{A} + \epsilon_0\mu_0\partial V/\partial t = 0$.)
(**a**) (15 points)
For this part, take C to be a (positive) constant, *i.e.* independent of time t as well as $\mathbf{r}$. Calculate the current density $\mathbf{J}$, flowing within this region, that produces $\mathbf{A}$. The *direction* and *sign* of your answer are important. (In this application, note that

$$\tfrac{4\pi}{\mu_0}\mathbf{A}(\mathbf{r}) \neq \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}d\tau' \ ,$$

because the current-carrying region is infinite in extent.)
Solution:
Combining Ampère's law with Griffiths' vector identity (11),

$$
\begin{aligned}
\mu_0\mathbf{J} &= \nabla \times \mathbf{B} \\
&= \nabla \times (\nabla \times \mathbf{A}) \\
&= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \\
&= 0 - \frac{1}{s}\frac{\partial}{\partial s}s\frac{\partial}{\partial s}\hat{z}\mu_0 C s^2 \\
\mathbf{J} &= -\hat{z}\,C\frac{1}{s}\frac{\partial}{\partial s}2s^2 \\
&= -\hat{z}\,4C \ .
\end{aligned}
$$

Notice that $\mathbf{A}$ and $\mathbf{J}$ point in opposite directions!
The above is the most direct path to the result. Alternatively, one may first evaluate $\mathbf{B}$:

$$
\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
&= -\hat{\phi}\frac{\partial A_z}{\partial s} \\
&= -\hat{\phi}\,2\mu_0 C s \ ,
\end{aligned}
$$

where the term in the middle equation includes the only nonvanishing derivative in the curl. Then

$$
\begin{aligned}
\mu_0\mathbf{J} &= \nabla \times \mathbf{B} \\
&= \frac{\hat{z}}{s}\frac{\partial}{\partial s}sB_\phi \\
&= -\frac{\hat{z}}{s}\frac{\partial}{\partial s}s2\mu_0 C s \\
\mathbf{J} &= -\hat{z}\,4C \ .
\end{aligned}
$$

(**b**) (20 points)
For this part, take C to be a decaying function of time, *i.e.*

$$C(t) = C_0 \exp\left(-t/\tau\right) \ ,$$

where C_0 and τ are positive constants. Consider a rectangular loop drawn at constant azimuth ϕ, bounded by $0 < z < z_0$ and $0 < s < s_0$. Calculate the EMF $\mathcal{E}$ around this loop (the sign of your answer won't be graded).
Solution:
The electric field is easily calculated from

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \ .$$

The potential term integrates to zero around the loop and thus plays no role. Because $\mathbf{A}$ is in the $\hat{z}$ direction and vanishes on the z axis, the only contribution to the integral comes from the outer segment where $s = s_0$ and $d\mathbf{l} = \hat{z}dz$. Proceeding counterclockwise around the loop,

$$\begin{aligned}
\mathcal{E} &= \oint \mathbf{E} \cdot d\mathbf{l} \\
&= -\oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} \\
&= -\int_{z_0}^{0} \frac{\partial}{\partial t} \hat{z} \mu_0 C_0 s_0^2 \exp\left(-t/\tau\right) \cdot \hat{z}dz \\
&= \int_{z_0}^{0} \frac{\mu_0 C_0 s_0^2 \exp\left(-t/\tau\right)}{\tau} dz \\
&= -\frac{\mu_0 C_0 s_0^2 z_0 \exp\left(-t/\tau\right)}{\tau} .
\end{aligned}$$

The above is the most direct path to the result. Alternatively, one may first calculate the magnetic flux Φ through the loop, then obtain $\mathcal{E}$ from its time derivative. This flux is most easily evaluated by performing the line integral of $\mathbf{A}$ around the loop. Again proceeding counterclockwise,

$$\begin{aligned}
\Phi &= \oint \mathbf{A} \cdot d\mathbf{l} \\
&= \int_{z_0}^{0} \mu_0 C_0 s_0^2 \exp\left(-t/\tau\right)dz \\
&= -\mu_0 s_0^2 z_0 C_0 \exp\left(-t/\tau\right) .
\end{aligned}$$

This same flux may also be obtained by integrating $\mathbf{B}$ from part (a). Proceeding counterclockwise around the loop, $d\mathbf{a}$ is in the $\hat{\phi}$ direction, opposite to the direction of $\mathbf{B}$. Therefore the flux is negative. Performing the integration,

$$\begin{aligned}
\mathbf{B} &= -\hat{\phi}\, 2\mu_0 C s \\
\Phi &= \int \mathbf{B} \cdot d\mathbf{a} \\
&= -\int_{0}^{s_0} ds \int_{0}^{z_0} dz\, 2\mu_0 C s \\
&= -\mu_0 s_0^2 z_0 C \\
&= -\mu_0 s_0^2 z_0 C_0 \exp\left(-t/\tau\right) .
\end{aligned}$$

With the same flux calculated either way, Fara-

day's law yields the EMF:

$$\begin{aligned}
\mathcal{E} &= -\frac{d\Phi}{dt} \\
&= -\frac{\mu_0 C_0 s_0^2 z_0 \exp\left(-t/\tau\right)}{\tau} .
\end{aligned}$$

(c) (15 points)
If you were asked to calculate the current density $\mathbf{J}$ for the conditions of part (b), where $\mathbf{A}$ decays with time, would you expect $\mathbf{J}$ to have the same dependence on s within our cylindrical region that you obtained in part (a)? Why or why not?
Solution:
Now that conditions are not static, Maxwell's corrected version of Ampère's Law is needed:

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} .$$

Though $\nabla \times \mathbf{B}$ has no s-dependence within our cylindrical region, the contribution of $d\mathbf{E}/dt$ to $\mathbf{J}$ is proportional to s^2, as is $\mathbf{A}$ itself. Therefore the Maxwell-corrected $\mathbf{J}$ will not have the same s-dependence as in part (a).

Problem 2. (50 points)
A nickel (five-cent coin) of radius a and thickness $d \ll a$ carries a uniform permanent magnetization

$$\mathbf{M} = \hat{z}M_0 ,$$

where M_0 is a positive constant and $\hat{z}$ is the nickel's axis of cylindrical symmetry.
(a) (30 points)
Calculate the magnetic field $\mathbf{B}(0,0,0)$ at the center of the nickel. The *direction* of $\mathbf{B}$ is important; express $\mathbf{B}$ to lowest nonvanishing order in d/a.
Solution:
The volume magnetization $\mathbf{M}$ yields a bound surface current $\mathbf{K}_b = \mathbf{M} \times \hat{n}$. Therefore $\mathbf{K}_b$ vanishes on the nickel's flat surfaces, and is equal to $\hat{\phi}M_0$ on its curved surface. A surface current on this thin curved strip $d \ll a$ is equivalent to a line current $I_b = K_b d$. Therefore $\mathbf{B}$ at the center is the same as the field from a circular loop.

Applying the Biot-Savart law,

$$\frac{4\pi}{\mu_0 I}d\mathbf{B}(\mathbf{r}=0) = \frac{d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{\hat{\phi}s d\phi \times (-\hat{s})}{s^2}$$

$$= \hat{z}\frac{d\phi}{a}$$

$$\mathbf{B}(0) = \hat{z}\frac{\mu_0 K_b d}{2a}$$

$$= \hat{z}\,\mu_0 M_0 \frac{d}{2a} \ .$$

(b) (20 points)
In the plane $z = 0$, draw counterclockwise a large circular loop $s = b \gg a$ that is centered on the nickel. What magnetic flux Φ flows through this loop? The *sign* of Φ is important; express Φ to lowest nonvanishing order in d/b.
Solution:
Far from the nickel, the field is that of a magnetic dipole with moment

$$\mathbf{m} = \hat{z}\, M_0 \pi a^2 d \ .$$

But the perfect-dipole approximation breaks down when we get close to the nickel, so it's tough to calculate Φ by integrating $\mathbf{B}$ over the loop's inner area.
The most straightforward approach uses the fact that the flux Φ through a loop is the integral of $\mathbf{A}$ around the loop; the dipole approximation for $\mathbf{A}$ will work well at the boundary of the loop, where $b \gg a$. First calculate $\mathbf{A}$:

$$\frac{4\pi}{\mu_0}\mathbf{A} = \frac{\mathbf{m} \times \hat{r}}{r^2}$$

$$= \frac{M_0\pi a^2 d}{r^2}\hat{z} \times (\hat{z}\cos\theta + \hat{s}\sin\theta)$$

$$= \frac{M_0\pi a^2 d}{b^2}\hat{z} \times \hat{s}$$

$$= \hat{\phi}\frac{M_0\pi a^2 d}{b^2}$$

$$\mathbf{A} = \hat{\phi}\frac{\mu_0 M_0 a^2 d}{4b^2} \ .$$

Since $\mathbf{A}$ is in the azimuthal direction, its line integral around the large circle is just $2\pi b A$, so

$$\Phi = \mu_0\pi a^2 M_0 \frac{d}{2b} \ .$$

Note that, as $b \to \infty$, all the flux through the nickel is returned within the large circle, so $\Phi \to 0$.
The above is the most direct path to the result. An alternative approach starts from the equation

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0 \ .$$

Choose a closed surface consisting of the plane $z = 0$ plus the hemispherical cap $r = \infty$. The cap makes no contribution to the integral because $\mathbf{B}$ from a dipole diminishes as r^{-3}. The plane can be divided into $s < b$ and $s > b$. Since the surface integral over the plane vanishes, the inner and outer portions give equal and opposite contributions. We evaluate the outer portion because the dipole approximation works well in that region.

$$\Phi = \int_0^b ds \int_0^{2\pi} s\,d\phi\,B_z$$

$$= -\int_b^{\infty} ds \int_0^{2\pi} s\,d\phi\,B_z \ .$$

In the plane $z = 0$, with $\hat{m} = \hat{z}$, the dipole's magnetic field is

$$\frac{4\pi r^3}{\mu_0 m}\mathbf{B} = 3(\hat{m}\cdot\hat{r})\hat{r} - \hat{m}$$

$$\frac{4\pi s^3}{\mu_0 m}\mathbf{B} = 3(\hat{z}\cdot\hat{s})\hat{s} - \hat{z}$$

$$= -\hat{z}$$

$$\mathbf{B} = -\hat{z}\frac{\mu_0 m}{4\pi s^3}$$

$$= -\hat{z}\frac{\mu_0\pi a^2 M_0 d}{4\pi s^3}$$

$$= -\hat{z}\frac{\mu_0 a^2 M_0 d}{4s^3} \ .$$

Performing the integral over the outer region,

$$\Phi = -\int_b^{\infty} ds \int_0^{2\pi} s\,d\phi\left(-\frac{\mu_0 a^2 M_0 d}{4s^3}\right)$$

$$= \frac{\mu_0 a^2 M_0 d}{4}2\pi\int_b^{\infty}\frac{ds}{s^2}$$

$$= \mu_0\pi a^2 M_0 \frac{d}{2b} \ .$$